PART : MATHEMATICS

1. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}=4\hat{i}+3\hat{j}+\hat{k}$ then the value of $\left((\vec{a}+\vec{b})\times(\vec{a}-(\vec{a}-\vec{b})\times\vec{b})\right)\times\vec{c}$ is :

(1) $(30\hat{i}-34\hat{j}+36\hat{k})$ (2) $(30\hat{i}+34\hat{j}+36\hat{k})$

(3) $(30\hat{i}+34\hat{j}-36\hat{k})$ (4) None of these

Ans. (1)

Sol. $\vec{a}+\vec{b}=3\hat{i}+2\hat{j}=\vec{r}_1$

$\vec{r}_2=\vec{a}-(\vec{a}-\vec{b})\times\vec{b}$ = $\vec{r}_2=(\hat{i}+\hat{j}+\hat{k})-(-\hat{i}+2\hat{k})\times(2\hat{i}+\hat{j}-\hat{k})$

$(\hat{i}+\hat{j}+\hat{k})-(-\hat{k}-\hat{j}+4\hat{j}-2\hat{i})=(3\hat{i}-2\hat{j}+2\hat{k})$

Now $\left((\vec{a}+\vec{b})\times(\vec{a}-(\vec{a}-\vec{b})\times\vec{b})\right)\times\vec{c}=(\vec{r}_1\times\vec{r}_2)\times\vec{c}$

$=(\vec{r}_1.\vec{c})\vec{r}_2-(\vec{r}_2.\vec{c})\vec{r}_1$

$=18(3\hat{i}-2\hat{j}+2\hat{k})-8(3\hat{i}+2\hat{j})$

$=(30\hat{i}-34\hat{j}+36\hat{k})$ Ans.

2 Which of the following is logically equivalent to $(p \vee q) \wedge (\sim p \rightarrow q)$ is :

(1) $(p \vee q)$ (2) $(p \wedge q)$

(3) $\sim p \vee q$ (4) $p \wedge \sim q$

Ans. (1)

Sol. We know $p \rightarrow q = \sim p \vee q$

Hence $\sim p \rightarrow q = p \vee q$

$\Rightarrow (p\vee q) \wedge (p\vee q) = p \vee q$ **Ans.**

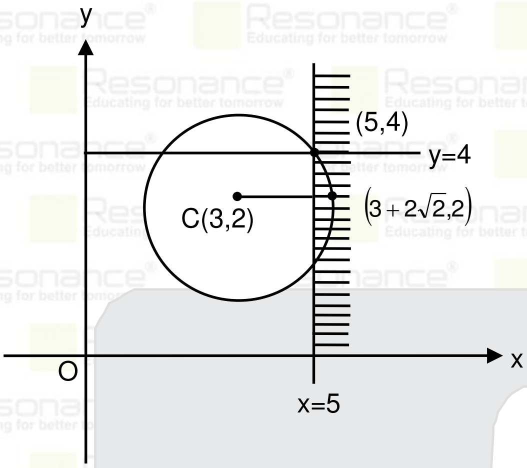
3. If $S_1 = \{z : |z-3-2i|^2 = 8\}$ and $S_2 = \{z : |z-\bar{z}| = 8\}$ and $S_3 = \{z : re(z) \geq 5\}$ then $S_1 \cap S_2 \cap S_3$ has

(1) Infinite many element (2) Only one element

(3) No any element (4) Two element

Ans. (2)

Sol. Let z = x+iy



$S_2 \Rightarrow |2iy| = 8 \Rightarrow |y| = 4$

$y = \pm 4$

$S_3 \Rightarrow x \geq 5$

$S_1 \Rightarrow |z-3-2i|^2 = 8 = \ (x-3)^2 + (y-2)^2 = 8$

Is a circle with centre (3,2) and radius = $2\sqrt{2}$

$\because$ Circle passes through (5,4)

$\Rightarrow$ There is exactly one point (5,4) in $S_1 \cap S_2 \cap S_3$

4. If $e^{-x}\int_3^x \{3t^2 + 2t + 4f'(t)\}dt = f(x)$ and $f'(4) = \frac{\alpha e^{\beta} - 224}{(e^{\beta} - 4)^2}$ then value of $(\alpha + \beta)$ is

Ans. **16.00**

Sol. Put x = 3 $\Rightarrow$ f(3) = 0

$e^{-x}[t^3 + t^2 + 4(F(t)_3^x] = F(x)$

$e^{-x}(x^3 + x^2 + 4F(x) - (27 + 9 + 4F(3) = F(x)$

$\Rightarrow F(x) = e^{-x}(x^3 + x^2 - 36 + 4F(x))$

$\Rightarrow e^x F(x) = x^3 + x^2 - 36 + 4F(x)$

$$F(x) = \frac{x^3 + x^2 - 36}{(e^x - 4)}$$

$$F'(x) = \frac{(3x^2 + 2x)(e^x - 4) - (e^x)(x^3 + x^2 - 36)}{(e^x - 4)^2}$$

$$F'(4) = \frac{56(e^4 - 4) - e^4(44)}{(e^4 - 4)^2}$$

$$F'(4) = \frac{12e^4 - 224}{(e^4 - 4)^2}$$

Hence $\alpha = 12$ and $\beta = 4$

$\alpha + \beta = 16$

5. The coefficient of x^7 and x^{-7} in the expansion of $\left(x^2+\frac{1}{bx}\right)^{11}$ and $\left(x+\frac{1}{bx^2}\right)^{11}$ respectively are equal then the value of b is :

(1) 1 (2) –1 (3) 0 (4) 2

Ans. **(1)**

Sol. $\left(x^2+\frac{1}{bx}\right)^{11}$

$T_{r+1} = {}^{11}C_r\,(x^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r\,(x)^{22-3r}\left(\frac{1}{b}\right)^r$

for $x^4 \Rightarrow 22 - 3r = 7 \Rightarrow r = 5$

$\therefore$ coefficient of $x^7 = {}^{11}C_5\left(\frac{1}{b}\right)^5$

similarly $\left(x+\frac{1}{bx^2}\right)^{11}$

$T_{r+1} = {}^{11}C_r\,(x)^{11-r}\left(\frac{1}{bx^2}\right)^r = {}^{11}C_r\left(\frac{1}{b}\right)^r(x)^{11-3r}$

For $x^{-7} \Rightarrow 11 - 3r = -7 \Rightarrow r = 6$

$\therefore$ coefficient $x^{-7} = {}^{11}C_6\left(\frac{1}{b}\right)^6$

${}^{11}C_6\left(\frac{1}{b}\right)^6 = {}^{11}C_5\left(\frac{1}{b}\right)^5 \Rightarrow b = 1$

6. Evaluate $\lim\limits_{x\to 2}\frac{x^2f(2)-4f(x)}{x-2}$, if f(2) = 4 & f'(2) = 1

Ans. (12)

Sol. $\lim\limits_{x\to 2}\frac{x^2f(2)-4f(x)}{x-2} = \lim\limits_{x\to 2}\frac{2xf(2)-4f'(x)}{1} = 2\cdot(2)\cdot f(2) - 4f'(2) = 16 - 4 = 12$

7. If $\sin\theta + \cos\theta = \frac{1}{2}$, then the value of $16(\sin2\theta + \cos4\theta + \sin6\theta)$ is :

(1) 23 (2) –23 (3) 36 (4) –43

Ans. (2)

Sol. $(\sin\theta + \cos\theta)^2 = \frac{1}{4}$

$1 + \sin2\theta = \frac{1}{4}$

$\sin2\theta = \frac{-3}{4}$

$16(\sin2\theta + \cos4\theta + \sin6\theta) = 16(\sin2\theta + 1 - 2\sin^2 2\theta + 3\sin2\theta - 4\sin^3 2\theta)$

$= 16(4\sin2\theta + 1 - 2\sin^2 2\theta - 4\sin^3 2\theta) = -23$

8. If mean and variance of eight observation 10, 13, 6, 7, a, 12, b, 12 are 9 and $\frac{37}{4}$ respectively, then the value of $(a-b)^2$ is s

(1)25 (2) 36 (3) 16 (4) 49

Ans. (3)

Sol. $9 = \frac{10+13+6+7+a+12+b+12}{8}$

$\Rightarrow a + b = 12$……… (i)

$\sum x_i^2 = 100+169+36+49+a^2+144+b^2+144$

$= a^2 + b^2 + 642$

$\sigma^2 = \frac{\sum x_i^2}{8} - (\bar{x})^2 = \frac{37}{4}$

$\frac{a^2+b^2+642}{8} - 81 = \frac{37}{4}$

$\Rightarrow a^2 + b^2 = 80 \Rightarrow a^2 + (12-a)^2 = 80 \Rightarrow a^2 - 12a + 32 = 0$

$\Rightarrow a = 4$ or $a = 8$

$b = 8$ or $b = 4$

$\Rightarrow (a-b)^2 = 16$ Ans.

9. If matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$ and $A^{-1} = \alpha I + \beta A$, $\alpha, \beta \in R$. Then the value of $(\alpha - 6\beta)$ is equal to :

(1) $\frac{3}{2}$ (2) $\frac{1}{2}$ (3) $\frac{-1}{2}$ (4) $\frac{-3}{2}$

Ans. (1)

Sol. Equation of $|A-\lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(1-\lambda) + 4 = 0 \Rightarrow \lambda^2 - 3\lambda + 2 + 4 = 0 \Rightarrow \lambda^2 - 3\lambda + 6 = 0$

$A^2 - 3A + 6I = 0$

Multiply both side by A^{-1}

$A - 3I + 6A^{-1} = 0$

$A^{-1} = \frac{-A}{6} + \frac{I}{2}$ ……………………(1)

$A^{-1} = \alpha I + \beta A$ ……………………(2)

Comparing equation (1) and (2)

$\alpha = \frac{1}{2}, \beta = \frac{-1}{6}$

$\alpha - 6\beta = \frac{1}{2} + 1 = \frac{3}{2}$

10. If α and β are the roots of the equation $x^2 - 20^{1/4}x + 5^{1/2} = 0$, then the value of $\alpha^8 + \beta^8$ is equal to :

(1) 25 (2) 50 (3) 75 (4) 100

Ans. **(2)**

Sol. $x^2 - 20^{1/4}x + 5^{1/2} = 0$

$(x^2 + \sqrt{5})^2 = \sqrt{20}\,x^2$

$x^4 + 5 + 2\sqrt{5}\,x^2 = 2\sqrt{5}\,x^2$

$x^4 = -5$

$x^8 = 25$

$\alpha^8 = 25, \beta^8 = 25$

$\alpha^8 + \beta^8 = 50$

11. Let n be a two digit natural number. If n is selected at random then find the probability such that $2^n - 2$ is a multiple of 3.

(1) $\frac{1}{4}$ (2) $\frac{1}{3}$ (3) $\frac{1}{2}$ (4) 0

Ans. **(3)**

Sol. Total number of cases = ${}^{90}C_1 = 90$

Now $2^n - 2 = (3-1)^n - 2$

$= {}^nC_0\, 3^n - {}^nC_1\, 3^{n-1} + ... + (-1)^{n-1}\, {}^nC_{n-1}\, 3 + (-1)^n\, {}^nC_n - 2$

$= 3(3^{n-1} - n.3^{n-2} + ... + (-1)^{n-1}.n) + (-1)^n - 2$

So $(2^n - 2)$ is multiple of 3 only when n is odd

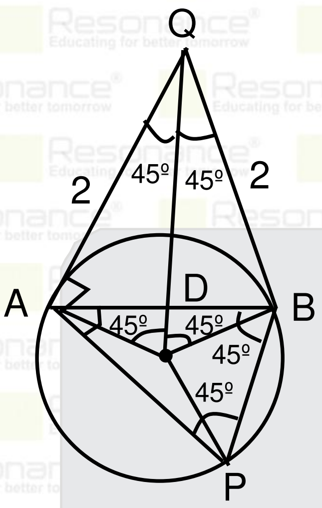
So number of favourable cases = 45, Hence required probability = $\frac{45}{90} = \frac{1}{2}$

12. If tangents to the circle $(x-1)^2 + (y-3)^2 = 2^2$ are drawn from the point $(-1, 1)$ such that the points of contact of tangents are A and B. Also a point P lies on the circle such that AB = AP, then the area of ΔABP is :

(1) 2 (2) $\sqrt{3}$ (3) 4 (4) $2\sqrt{2}$

Ans. (3)

Sol.



$AQ = \sqrt{S_1} = \sqrt{(-1-1)^2 + (1-3)^2 - 4} = 2$

AD = 2sin45º

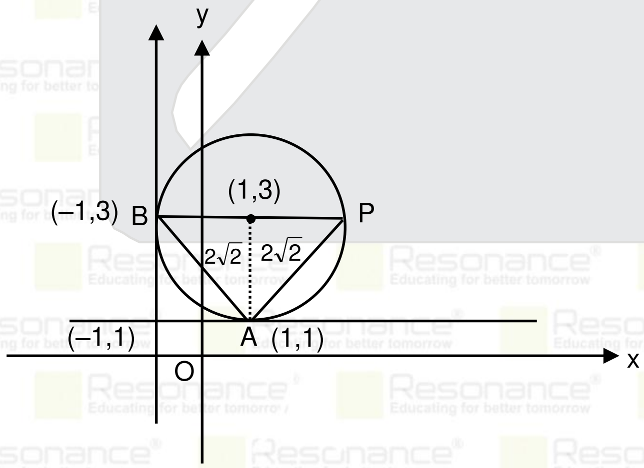
AB = 4sin45º

AP = 4sin45º

Area of triangle ABP = $\frac{1}{2}$ (4sin45º) (4sin45º)

= 4

Alternate :



$\Delta PAB = \frac{1}{2} \times 4 \times 2 = 4$

13. If $\log_3 2$, $\log_3(2^x - 5)$, $\log_3\left(2^x - \frac{7}{2}\right)$ are in A.P. then the value of x is

(1) 4 (2) 2 (3) 3 (4) 5

Ans. (3)

Sol. $2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \log_3(2^x - 5)^2 = \log_3\left(2\left(2^x - \frac{7}{2}\right)\right)$

$\Rightarrow (2^x)^2 - 10(2^x) + 25 = 2(2^x) - 7$

$\Rightarrow (2^x)^2 - 12(2^x) + 32 = 0$

$\Rightarrow 2^x = 4$ or 8

$\Rightarrow x = 2$ or 3

For $x = 2$, $\log(2^x - 5)$ is not defined

$\Rightarrow x = 3$

14. If $\sec y \frac{dy}{dx} = \sin(x+y) + \sin(x-y)$; $y(0) = 0$ then the value of $5y'\left(\frac{\pi}{2}\right)$ is

(1) 3 (2) 4 (3) 2 (4) 1

Ans. **(3)**

Sol. $\sec y \frac{dy}{dx} = 2\sin(x)\cos(y)$ ....(1)

$\int \sec^2 y \, dy = \int 2\sin x \, dx$

$\tan y = -2\cos x + c$

When $x = 0$, $y = 0 \Rightarrow c = 2$

$\tan y = -2\cos x + 2$ ....(2)

$\because \sec^2 y \frac{dy}{dx} = 2\sin x$

$(1 + \tan^2 y) \frac{dy}{dx} = 2\sin x$ ....(3)

By equation (2) & (3)

$\frac{dy}{dx} = \frac{2\sin x}{1 + (2 - 2\cos x)^2}$

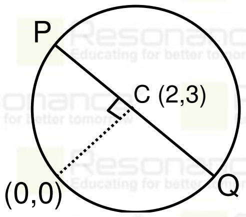
$f'\left(\frac{\pi}{2}\right) = \frac{2}{1 + (2)^2} = \frac{2}{5} \Rightarrow 5f'\left(\frac{\pi}{2}\right) = 2$

15. A circle with centre (2, 3) passes through the origin. A diameter PQ is such that it is perpendicular to the line joining the centre of circle and origin. The coordinates of points P and Q are :

(1) (–1, 5) & (5, 1) (2) (–2, 5) & (–5, 1) (3) (3, 6) & (5, 1) (4) (–1, 1) & (5, –1)

Ans. (1)

Sol. $(x-2)^2 + (y-3)^3 = 13$



$y = \frac{3}{2}x$

line perpendicular to the above line and passing through (2,3) is 3y +2x = 13

Coordinates of P,Q $\Rightarrow \left(2 \pm \sqrt{13}\cos\theta, 3 \pm \sqrt{13}\sin\theta\right)$

$\Rightarrow \left(2 \pm \sqrt{13}\left(\frac{-3}{\sqrt{13}}\right), 3 \pm \sqrt{13}\left(\frac{2}{\sqrt{13}}\right)\right)$

$\Rightarrow$ (–1, 5) & (5, 1)

16. Evaluate $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{j=1}^{n}\frac{((2j-1)+8n)}{((2j-1)+4n)}$

(1) $1+2\ln\left(\frac{3}{2}\right)$ (2) $1-2\ln\left(\frac{3}{2}\right)$ (3) $2\ln\left(\frac{3}{2}\right)-1$ (4) $3\ln\left(\frac{2}{3}\right)+5$

Ans. **(1)**

Sol. $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{j=1}^{n}\frac{\left(2\frac{j}{n}-\frac{1}{n}+8\right)}{\left(2\frac{j}{n}-\frac{1}{n}+4\right)}.\frac{1}{n}$

$= \int_0^1 \frac{2x+8}{2x+4}dx$

$= \int_0^1 dx + \int \frac{4}{2x+4}dx$

$= 1+4.\frac{1}{2}\left[\ln(2x+4)\right]_0^1$

$= 1+2\left[\ln 6-\ln 4\right]$

$= 1+2\ln\left(\frac{3}{2}\right)$

17. If $f(x) = \begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}} & -\frac{\pi}{4} < x < 0 \\ b & x = 0 \\ e^{\frac{\cot(4x)}{\cot(2x)}} & 0 < x < \frac{\pi}{4} \end{cases}$ is continuous, then find the value of ($a + \ell nb$).

Ans. **(1)**

Sol. RHL = $f(0^+) = \lim_{x \to 0^+} e^{\frac{\cot 4x}{\cot 2x}} = \lim_{x \to 0^+} e^{\frac{\tan 2x}{\tan 4x}} = e^{\frac{1}{2}}$

LHL = $f(0^-) = \lim_{x \to 0^-} (1 - \sin x)^{\frac{a}{-\sin x}}$

$= e^{\lim_{x \to 0^-} \frac{-a}{\sin x}(-\sin x)} = e^a$

For f(x) to be continuous

$f(0^+) = f(0^-) = f(0)$

$\Rightarrow \; e^{\frac{1}{2}} = e^a = b \;\Rightarrow\; a = \frac{1}{2}$ & $b = e^{\frac{1}{2}} \;\Rightarrow\; \ell nb = \frac{1}{2}$

$\Rightarrow \; a + \ell nb = 1$

18. Evaluate $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{x\cos x})(\sin^4 x + \cos^4 x)}$

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{2\sqrt{2}}$ (3) $\frac{\pi}{\sqrt{2}}$ (4) $\frac{\pi}{4}$

Ans. **(2)**

Sol. $f(a + b - x) = f(x)$

$I = \int_{-\pi/4}^{\pi/4} \frac{e^{x\cos x}}{(1+e^{x\cos x})(\sin^4 x + \cos^4 x)} dx$

$2I = \int_{-\pi/4}^{\pi/4} \frac{1}{\sin^4 x + \cos^4 x}$

$2I = \int_{-\pi/4}^{\pi/4} \frac{1}{1 - \frac{1}{2}\sin^2 2x} dx$

$2I = \int_{-\pi/4}^{\pi/4} \frac{2\sec^2 2x}{\sec^2 2x + 1} dx$ (put tan2x = t

$2I = \frac{4}{2}\int_0^{\infty} \frac{dt}{2+t^2} = \left(\frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}}\right)_0^{\infty}$

$I = \frac{\pi}{2\sqrt{2}}$

19. If the domain of the function f(x) = $\log_5(\log_4(\log_3(18x-x^2-77)))$ is (a,b), then find the value of $\int_a^b \frac{\sin^3 x\,dx}{\sin^3 x+\sin^3(a+b-x)}$

Ans. **(1)**

Sol. $\log_4 \log_3 (18x - x2 - 77) > 0$

$18x - x^2 - 77 > 3$

$x^2 - 18x + 80 < 0$

$(x-8)(x-10) < 0$

$x \in (8, 10)$

$\therefore$ a = 8 and b = 10

$$\therefore I = \int_a^b \frac{\sin^3 x\,dx}{\sin^3 x+\sin^3(a+b-x)} = \int_8^{10} \frac{\sin^3 x\,dx}{\sin^3 x+\sin^3(a+b-x)}$$

Using the property

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$$2I = \int_8^{10} 1dx = 2$$

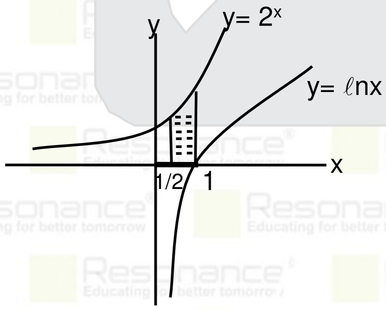
$\therefore I = 1$ Ans.

20. Find area bounded by y = max {0, ℓnx} and y < 2^x where $\frac{1}{2} < x < 1$:

(1) $\frac{2+\sqrt{2}}{\ell n2}$ (2) $\frac{2-\sqrt{2}}{\ell n2}$ (3) $\frac{1-\sqrt{2}}{\ell n2}$ (4) $\frac{1+\sqrt{2}}{\ell n2}$

Ans. (2)

Sol.



Area is = $\int_{\frac{1}{2}}^{1} (2^x - 0)dx = \left(\frac{2^x}{\ell n2}\right)_{\frac{1}{2}}^{1} = \frac{2-\sqrt{2}}{\ell n2}$

21. If f(x) = $\begin{vmatrix} \sin^2 x & -2+\cos^2 x & \cos 2x \\ 2+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix}$, $x \in (0, \pi]$. Then the maximum value of f(x) is ?

Ans. **0**6.00

Sol. $R_1 \to R_1 - R_2$

$R_2 \to R_2 - R_3$

$= \begin{vmatrix} -2 & -2 & 0 \\ 2 & 0 & -1 \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix}$

$= -2(\cos^2 x) + 2(2+2\cos 2x + \sin^2 x)$

$= -2\cos^2 + 4 + 4\cos 2x + 2\sin^2 x$

$= -2(\cos^2 x - \sin^2 x) + 4\cos 2x + 4$

$= 4 + 2\cos 2x = 2(2+\cos 2x)$, maximum value of f(x) = 6